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A 2-BASIC SET OF THE ALTERNATING GROUP

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Abstract. In this note, we construct a 2-basic set of the alternating group $\mathfrak{A}_n$. To do this, we construct a 2-basic set of the symmetric group $\mathfrak{S}_n$ with an additional property, such that its restriction to $\mathfrak{A}_n$ is a 2-basic set. We adapt here a method developed in [2] for the case when the characteristic is odd. One of the main tools is the generalized perfect isometries defined by Külshammer, Olsson and Robinson in [9].

1. Introduction

This note is concerned with the existence problem of basic sets for finite groups. Let G be a finite group and p be a prime. A p-basic set of G is a subset B of the set $\mathrm{Irr}(G)$ of irreducible complex characters of G, such that the restrictions $B^{p\text{-reg}} = \{\chi^{p\text{-reg}},\, \chi \in B\}$ to p-regular elements (i.e., the elements of G with order prime to p) of the characters in B form a $\mathbb{Z}$-basis of the ring of p-Brauer characters of G. Gerhard Hiss has conjectured that every finite group has a p-basic set. It is actually proved for some finite groups (see for example [5], [3], [4] and [1]), but it remains open and a difficult question in general.

Recently, the authors proved in [2] that the alternating group $\mathfrak{A}_n$ has a p-basic set for any odd prime p. In this note, we complete this work by constructing a 2-basic set of $\mathfrak{A}_n$.

Recall that the irreducible characters of the symmetric group $\mathfrak{S}_n$ are naturally labelled by the partitions of n [8, 2.1.11]. We denote by $\mathcal{P}$ (respectively $\mathcal{P}_n$) the set of all partitions of all integers (respectively of n). For any $\lambda \in \mathcal{P}$, we write $|\lambda|$ for the size of λ. For any partition λ of n (written $\lambda \vdash n$ in the following), we denote by χ_λ the corresponding irreducible character of $\mathfrak{S}_n$ and by λ^* the conjugate partition of λ. Then $\chi_{\lambda^*} = \varepsilon\chi_\lambda$, where ε is the signature of $\mathfrak{S}_n$. A partition λ is said to be *self-conjugate* if $\lambda = \lambda^*$.

For any $B \subseteq \mathrm{Irr}(\mathfrak{S}_n)$, we call *restriction of B to $\mathfrak{A}_n$* the subset $B_{\mathfrak{A}_n}$ of $\mathrm{Irr}(\mathfrak{A}_n)$ consisting of all the irreducible constituents of the restrictions to $\mathfrak{A}_n$ of the characters in B. To prove the existence of a p-basic set of $\mathfrak{A}_n$ for odd prime p, the approach in [2] consists in constructing a p-basic set $\mathcal{B}$ of $\mathfrak{S}_n$ satisfying two additional properties which ensure that $\mathcal{B}_{\mathfrak{A}_n}$ is a p-basic set of $\mathfrak{A}_n$ (cf. [2, 1.1]).

We will see that, when $p = 2$, we can use a similar strategy, by requiring the single additional property that $\mathcal{B}$ contains all characters labelled by self-conjugate partitions; namely, we will prove

Proposition 1.1. *If B is a 2-basic set of $\mathfrak{S}_n$ containing every $\chi_\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ with $\lambda = \lambda^*$, then the restriction $B_{\mathfrak{A}_n}$ of B to $\mathfrak{A}_n$ is a 2-basic set of $\mathfrak{A}_n$.*

The aim of this paper is to adapt the methods of [2] (some of which heavily relie on p being odd) to the case $p = 2$, in order to construct a 2-basic set of $\mathfrak{S}_n$

satisfying Proposition 1.1.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of n. We denote by $\alpha_\lambda = (\lambda^{(1)}, \lambda^{(2)})$ the 2-quotient of λ. Note that this is only defined up to a choice of convention. However, it is proved in [2] (cf. the proof of Lemma 3.1) that, for a certain choice, we have that, for any n and any $\lambda \vdash n$, if $\alpha_\lambda = (\lambda^{(1)}, \lambda^{(2)})$, then $\alpha_{\lambda^*} = (\lambda^{(2)^*}, \lambda^{(1)^*})$. From now on, we always assume this to be the case. Then our main result is

Theorem 1.2. *For every integer m, we set*

$$\mathcal{P}'_m = \{\lambda = (\lambda_1, \ldots, \lambda_r) \vdash m \mid \lambda_i \text{ is even for } 1 \leq i \leq r\}.$$

Define

$$\Lambda = \{(\mu, \emptyset) \mid \mu \notin \mathcal{P}'_{|\mu|}\} \cup \{(\mu, \mu^*),\, \mu \in \mathcal{P}\}.$$

Then the set $\mathcal{B}_\Lambda$ of irreducible characters χ_λ of $\mathfrak{S}_n$ satisfying $\alpha_\lambda \in \Lambda$ is a 2-basic set of $\mathfrak{S}_n$. Moreover the restriction $\mathcal{B}_{\Lambda,\mathfrak{A}_n}$ of $\mathcal{B}_\Lambda$ to $\mathfrak{A}_n$ is a 2-basic set of $\mathfrak{A}_n$.

Throughout this article, we will use the following notations and conventions. For G a finite group, $\operatorname{Irr}(G)$ denotes the set of complex irreducible characters of G. For ϕ_1, $\phi_2 \in \operatorname{Irr}(G)$, we denote by $\phi_1 \otimes \phi_2$ the character defined by $\phi_1 \otimes \phi_2(g) = \phi_1(g)\phi_2(g)$. Let H and K be two finite groups. For $\phi_H \in \operatorname{Irr}(H)$ and $\phi_K \in \operatorname{Irr}(K)$, we define $\phi_H \boxtimes \phi_K \in \operatorname{Irr}(H \times K)$ by

$$\phi_H \boxtimes \phi_K(h,k) = \phi_H(h)\phi_K(k) \quad \text{for } h \in H,\, k \in K.$$

For c a conjugacy class of G and χ a character of G, we sometimes will use the notation $\chi(c)$ for $\chi(g)$ for $g \in c$.

The paper is organized as follows. In Section 2, we prove Proposition 1.1. In Section 3.2, we construct, for each integer w, a $\mathbb{Z}$-basis of the ring $\mathbb{Z}\operatorname{Irr}(\mathfrak{S}_{2w})$ of virtual characters of $\mathfrak{S}_{2w}$ that we will need in order to prove Theorem 1.2. Finally, in Section 3, we prove Theorem 1.2.

2. Restriction to $\mathfrak{A}_n$ of 2-basic sets of $\mathfrak{S}_n$

We first prove the following lemma.

Lemma 2.1. *With the above notation, if $\lambda \vdash n$, then $\chi_\lambda^{2\text{-reg}} = \chi_{\lambda^*}^{2\text{-reg}}$.*

Proof. Let σ be an element of $\mathfrak{S}_n$ with odd order. Write $\sigma = c_1 \cdots c_r$ as product of disjoint cycles. Since the order of σ is odd, each of the c_i's must have odd length, so that $\varepsilon(\sigma) = 1$. Since $\chi_{\lambda^*} = \varepsilon\chi_\lambda$, this yields the claim. □

2.1. **Irreducible characters of $\mathfrak{A}_n$.** The construction and the values of the irreducible characters of $\mathfrak{A}_n$ are described in [8, 2.5.13]. For the convenience of the reader, we briefly recall how to parametrize them from those of $\mathfrak{S}_n$. Take any $\lambda \vdash n$, and let

$$\rho_\lambda := \operatorname{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\chi_\lambda).$$

If $\lambda \neq \lambda^*$, then $\rho_\lambda = \rho_{\lambda^*}$ is irreducible, and $\operatorname{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\rho_\lambda) = \chi_\lambda + \chi_{\lambda^*}$. Otherwise, ρ_λ is the sum of two irreducible characters of $\mathfrak{A}_n$, written $\rho_{\lambda,\pm}$, and chosen as follows. If $\lambda = \lambda^* = (\lambda_1, \ldots, \lambda_r) \vdash n$, then we let $\overline{\lambda} = (2\lambda_1 - 1, 2\lambda_2 - 3, \ldots, 1) \vdash n$.The conjugacy class of $\mathfrak{S}_n$ of cycle type $\overline{\lambda}$ consists of elements of $\mathfrak{A}_n$, and splits into two classes $\overline{\lambda}_\pm$ of $\mathfrak{A}_n$. Now, if $x \in \mathfrak{A}_n$ doesn't have cycle type $\overline{\lambda}$, then $\rho_{\lambda,+}(x) = \rho_{\lambda,-}(x)$. If $x_\pm \in \overline{\lambda}_\pm$, then $\rho_{\lambda,\pm}(x_+) = s_\lambda \pm t_\lambda$ and $\rho_{\lambda,\pm}(x_-) = s_\lambda, \mp t_\lambda$, with s_λ and t_λ as described in [8, 2.5.13]. Furthermore, $\operatorname{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\rho_{\lambda,+}) = \operatorname{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\rho_{\lambda,-}) = \chi_\lambda$.

2.2. **Proof of Proposition 1.1.** The proof of Proposition 1.1 is based on the proof of [2, 5.2]. Suppose that $\mathcal{B}$ is a 2-basic set of $\mathfrak{S}_n$ as in Proposition 1.1 and consider its restriction $\mathcal{B}_{\mathfrak{A}_n}$ to $\mathfrak{A}_n$. To prove that $\mathcal{B}_{\mathfrak{A}_n}$ is a 2-basic set of $\mathfrak{A}_n$ we have to show that $\mathcal{B}_{\mathfrak{A}_n}^{2\text{-reg}} = \{\chi^{2\text{-reg}} | \chi \in \mathcal{B}_{\mathfrak{A}_n}\}$ is free and generates over $\mathbb{Z}$ the ring $\mathbb{Z}\operatorname{Irr}(\mathfrak{A}_n)^{2\text{-reg}}$.

We denote by $\mathcal{S}$ the set of self-conjugate partition of n and by $\mathcal{T}$ the set of partitions of n such that $\lambda \in \mathcal{T}$ if and only if $\chi_\lambda \in \mathcal{B}$. Put $\mathcal{S}' = \mathcal{T} \backslash \mathcal{S}$. With this notation, we have

$$\mathcal{B}_{\mathfrak{A}_n} = \{\rho_\lambda \mid \lambda \in \mathcal{S}'\} \cup \{\rho_{\lambda,\pm} \mid \lambda \in \mathcal{S}\}.$$

Suppose that there are integers a_λ ($\lambda \in \mathcal{S}'$) and $b_{\lambda,\pm}$ ($\lambda \in \mathcal{S}$) such that

$$\sum_{\lambda \in \mathcal{S}'} a_\lambda \rho_\lambda^{2\text{-reg}} + \sum_{\lambda \in \mathcal{S}} (b_{\lambda,-}\rho_{\lambda,-}^{2\text{-reg}} + b_{\lambda,+}\rho_{\lambda,+}^{2\text{-reg}}) = 0. \tag{1}$$

For $\lambda \in \mathcal{S}'$, we have $\lambda^* \notin \mathcal{S}'$. Indeed, since $\chi_\lambda^{2\text{-reg}} = \chi_{\lambda^*}^{2\text{-reg}}$ (see Lemma 2.1), we cannot have $\chi_\lambda \in \mathcal{B}$ and $\chi_{\lambda^*} \in \mathcal{B}$ simultaneously because $\mathcal{B}^{2\text{-reg}}$ is free. Hence, for $\lambda \in \mathcal{S}'$, there is no $\lambda' \in \mathcal{S}'$ satisfying

$$\langle \operatorname{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\rho_\lambda), \operatorname{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\rho_{\lambda'}) \rangle_{\mathfrak{S}_n} \neq 0.$$

Moreover, $\operatorname{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\rho_\lambda) = \chi_\lambda + \chi_{\lambda^*}$ implies

$$\operatorname{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\rho_\lambda^{2\text{-reg}}) = 2\chi_\lambda^{2\text{-reg}}.$$

(Note that this holds because, for any class function α of $\mathfrak{A}_n$, we have $\operatorname{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\alpha^{2\text{-reg}}) = (\operatorname{Ind}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\alpha))^{2\text{-reg}}$.) Therefore, inducing Relation (1) from $\mathfrak{A}_n$ to $\mathfrak{S}_n$ we deduce

$$\sum_{\lambda \in \mathcal{S}'} 2a_\lambda \chi_\lambda^{2\text{-reg}} + \sum_{\lambda \in \mathcal{S}} (b_{\lambda,-} + b_{\lambda,+})\chi_\lambda^{2\text{-reg}} = 0.$$

But $\mathcal{B}^{2\text{-reg}}$ is free, implying $a_\lambda = 0$ for $\lambda \in \mathcal{S}'$ and $b_{\lambda,+} + b_{\lambda,-} = 0$. Relation (1) gives

$$\sum_{\lambda \in \mathcal{S}} b_{\lambda,+}(\rho_{\lambda,+}^{2\text{-reg}} - \rho_{\lambda,-}^{2\text{-reg}}) = 0.$$

Now, using the fact that $\rho_{\lambda,+}$ and $\rho_{\lambda,-}$ only differ on the conjugacy classes labelled by $\overline{\lambda}_+$ and $\overline{\lambda}_-$ we deduce that $\mathcal{B}_{\mathfrak{A}_n}^{2\text{-reg}}$ is free (we use here the same argument as in the proof of [2, 5.2]).

We now prove that $\mathcal{B}_{\mathfrak{A}_n}^{2\text{-reg}}$ generates $\mathbb{Z}\operatorname{Irr}(\mathfrak{A}_n)^{2\text{-reg}}$ over $\mathbb{Z}$. Let ρ be a character of $\mathfrak{A}_n$ which does not belong to $\mathcal{B}_{\mathfrak{A}_n}$. The definition of $\mathcal{B}_{\mathfrak{A}_n}$ implies that there is $\lambda \vdash n$ with $\lambda \neq \lambda^*$, such that $\rho = \rho_\lambda$. In particular, as explained in §2.1 we have

$$\rho_\lambda = \operatorname{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(\chi_\lambda).$$

Since $\mathcal{B}$ is a 2-basic set of $\mathfrak{S}_n$, there exist integers $\{a_\chi,\, \chi \in \mathcal{B}\}$ satisfying

$$\chi_\lambda^{2\text{-reg}} = \sum_{\chi \in \mathcal{B}} a_\chi \chi^{2\text{-reg}}.$$

Restricting this last relation to $\mathfrak{A}_n$, we see that $\rho_\lambda^{2\text{-reg}}$ is a $\mathbb{Z}$-linear combination of elements of $B_{\mathfrak{A}_n}^{2\text{-reg}}$. This yields the claim.

3. A 2-basic set of $\mathfrak{A}_n$

3.1. **A result on wreath products with cyclic kernel.** Throughout this section, we fix ℓ and w positive integers and put

$$G_{\ell,w} = \mathbb{Z}_\ell \wr \mathfrak{S}_w,$$

where $\mathbb{Z}_\ell$ denotes a cyclic group of order ℓ. We denote by ω a generator of $\mathbb{Z}_\ell$ and set $\operatorname{Irr}(\mathbb{Z}_\ell) = \{\psi_i \mid i = 1, \ldots, \ell\}$, with the convention that ψ_1 is the trivial character of $\mathbb{Z}_\ell$. In the following, we denote by $\mathcal{MP}_{\ell,w}$ the set of ℓ-tuples of partitions $(\mu_1, \ldots, \mu_\ell)$ such that $\sum |\mu_i| = w$.

We recall that the conjugacy classes of $G_{\ell,w}$ are parametrized by the elements of $\mathcal{MP}_{\ell,w}$ as follows. The elements of $G_{\ell,w}$ are of the form (h, σ) with $h = (h_1, \ldots, h_w) \in \mathbb{Z}_\ell^w$ and $\sigma \in \mathfrak{S}_w$. For any k-cycle $\kappa = (j, \kappa j, \ldots, \kappa^{k-1} j)$ in σ, we define

$$g((h, \sigma); \kappa) = h_j h_{\kappa j} \cdots h_{\kappa^{k-1} j} \in \mathbb{Z}_\ell.$$

Let $\sigma = \prod_{c \in s(\sigma)} c$ be the cycle structure of σ. We then form the corresponding ℓ-tuples of partitions $(\mu_1, \ldots, \mu_\ell)$ by adding a k-cycle to μ_i whenever $c \in s(\sigma)$ is a k-cycle satisfying $g((h, \sigma), c) = \omega^{i-1}$. The resulting ℓ-tuple $(\mu_1, \ldots, \mu_\ell)$ lies in $\mathcal{MP}_{\ell,w}$ and is the so-called *cycle structure* of (h, σ). Two elements of $G_{\ell,w}$ are conjugate if and only if they have the same cycle structure.

We define

$$\mathcal{C}_\emptyset = \{(\mu_1, \emptyset, \ldots, \emptyset) \mid \mu_1 \vdash w\}.$$

Remark 3.1. *Using $\sigma \mapsto (1, \sigma)$, we can identify $\mathfrak{S}_w$ to a subgroup of $G_{\ell,w}$. Note that $\sigma \in \mathfrak{S}_w$ is in the class of $\mathfrak{S}_w$ labelled by the partition $\mu_1 \vdash w$ if and only if $(1, \mu_1)$ lies in the class of $G_{\ell,w}$ with cycle structure $(\mu_1, \emptyset, \ldots, \emptyset) \in \mathcal{C}_\emptyset$.*

We recall that the irreducible characters of $G_{\ell,w}$ are also labelled by the elements of $\mathcal{MP}_{\ell,w}$ as follows. Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_\ell) \in \mathcal{MP}_{\ell,w}$. Consider the character

$$\phi_{\boldsymbol{\mu}} = \prod_{i=1}^{\ell} \underbrace{\psi_i \boxtimes \ldots \boxtimes \psi_i}_{|\mu_i| \text{ times}}. \tag{2}$$

If $I_{G_{\ell,w}}(\phi_{\boldsymbol{\mu}})$ denotes the inertial subgroup of $\phi_{\boldsymbol{\mu}} \in \operatorname{Irr}(\mathbb{Z}_\ell^w))$ in $G_{\ell,w}$, then

$$I_{G_{\ell,w}}(\phi_{\boldsymbol{\mu}}) = \mathbb{Z}_\ell^w \rtimes \prod_{i=1}^{\ell} \mathfrak{S}_{|\mu_i|} = \prod_{i=1}^{\ell} \mathbb{Z}_\ell \wr \mathfrak{S}_{|\mu_i|}.$$

Moreover, $\phi_{\boldsymbol{\mu}}$ can be extended to an irreducible character $\widehat{\phi}_{\boldsymbol{\mu}} = \boxtimes_{i=1}^{\ell} \widehat{\psi_i^{|\mu_i|}}$ of $I_{G_{\ell,w}}(\phi_{\boldsymbol{\mu}})$ by setting $\widehat{\phi}_{\boldsymbol{\mu}}(h, x) = \phi_{\boldsymbol{\mu}}(h)$ for $h \in \mathbb{Z}_\ell^w$ and $x \in \prod \mathfrak{S}_{|\mu_i|}$. The irreducible character $\theta_{\boldsymbol{\mu}}$ corresponding to $\boldsymbol{\mu} \in \mathcal{MP}_{\ell,w}$ is then given by

$$\theta_{\boldsymbol{\mu}} = \operatorname{Ind}_{I_{G_{\ell,w}}(\phi_{\boldsymbol{\mu}})}^{G_{\ell,w}} (\widehat{\phi_{\boldsymbol{\mu}}} \otimes (\chi_{\mu_1} \boxtimes \cdots \boxtimes \chi_{\mu_\ell})) = \operatorname{Ind}_{I_{G_{\ell,w}}(\phi_{\boldsymbol{\mu}})}^{G_{\ell,w}} \left(\prod_{i=1}^{\ell} \widehat{\psi_i^{|\mu_i|}} \otimes \chi_{\mu_i} \right), \tag{3}$$

where χ_{μ_i} denotes the irreducible characters of $\mathfrak{S}_{|\mu_i|}$ corresponding to the partition μ_i of $|\mu_i|$.

Proposition 3.2. *For $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_\ell) \in \mathcal{MP}_{\ell,w}$, write $\mathfrak{S}_{\boldsymbol{\mu}} = \mathfrak{S}_{\mu_1} \times \ldots \times \mathfrak{S}_{\mu_\ell}$ for the corresponding Young subgroup of $\mathfrak{S}_w$. Define $\theta_{\boldsymbol{\mu}}$ as in Formula (3) and put*

$$\Gamma_{\boldsymbol{\mu}} = \operatorname{Ind}_{\mathfrak{S}_{\boldsymbol{\mu}}}^{\mathfrak{S}_w} (\chi_{\mu_1} \boxtimes \cdots \boxtimes \chi_{\mu_\ell}),$$

where χ_{μ_i} *denotes the irreducible character of* $\mathfrak{S}_{|\mu_i|}$ *corresponding to* $\mu_i \vdash |\mu_i|$*. Then, for any* $\pi \vdash w$*, we have*

$$\theta_{\boldsymbol{\mu}}((\pi, \emptyset, \ldots, \emptyset)) = \Gamma_{\boldsymbol{\mu}}(\pi).$$

Proof. Let $\pi \vdash w$. We fix $x \in G_{\ell,w}$ in the conjugacy class of $G_{\ell,w}$ labelled by $(\pi, \emptyset, \ldots, \emptyset)$. Using [8, 4.2.10], we deduce

$$|\,\mathrm{C}_{G_{\ell,w}}(x)| = \ell^w \prod_k a_{1k}(x)! k^{a_{1k}(x)},$$

where $a_{1k}(x)$ denotes the number of k-parts of the first partition of the cycle structure of x. Denote by $a_k(\pi)$ the number of k-parts of π. Then we have $a_{1k}(x) = a_k(\pi)$ and [8, 1.2.15] implies

$$|\,\mathrm{C}_{G_{\ell,w}}(x)| = \ell^w |\,\mathrm{C}_{\mathfrak{S}_w}(\pi)|. \tag{4}$$

Note that

$$\mathbb{Z}_\ell^w \rtimes \mathfrak{S}_{\boldsymbol{\mu}} = \prod_{i=1}^{\ell} \mathbb{Z}_\ell \wr \mathfrak{S}_{|\mu_i|}.$$

Then, if we suppose that $\pi = (\pi_1, \ldots, \pi_l) \in \mathfrak{S}_{\boldsymbol{\mu}}$, it follows that

$$\mathrm{C}_{\mathbb{Z}_\ell^w \rtimes \mathfrak{S}_{\boldsymbol{\mu}}}(x) = \prod_{i=1}^{\ell} \mathrm{C}_{\mathbb{Z}_\ell \wr \mathfrak{S}_{|\mu_i|}}(\pi_i, \emptyset, \ldots, \emptyset).$$

Furthermore, applying Formula (4) with $(\pi_i, \emptyset, \ldots, \emptyset) \in G_{\ell,|\mu_i|}$, we deduce

$$\begin{aligned} |\,\mathrm{C}_{\mathbb{Z}_\ell^w \rtimes \mathfrak{S}_{\boldsymbol{\mu}}}(x)| &= \prod_{i=1}^{\ell} \ell^{|\mu_i|} |\,\mathrm{C}_{\mathfrak{S}_{|\mu_i|}}(\pi_i)| \\ &= \ell^w \prod_{i=1}^{\ell} |\,\mathrm{C}_{\mathfrak{S}_{|\mu_i|}}(\pi_i)| \\ &= \ell^w |\,\mathrm{C}_{\mathfrak{S}_{\boldsymbol{\mu}}}(\pi)|. \end{aligned}$$

Therefore, the induction formula for characters gives

$$\theta_{\boldsymbol{\mu}}(x) = |\,\mathrm{C}_{G_{\ell,w}}(x)| \sum_{i \in I} \frac{1}{|\,\mathrm{C}_{\mathbb{Z}_\ell^w \rtimes \mathfrak{S}_{\boldsymbol{\mu}}}(x_i)|} \left(\widehat{\phi}_{\boldsymbol{\mu}} \otimes (\chi_{\mu_1} \boxtimes \cdots \boxtimes \chi_{\mu_\ell})\right)(x_i),$$

where $\widehat{\phi}_{\boldsymbol{\mu}}$ is defined in Equation (2) and $\{x_i,\, i \in I\}$ is a system of representatives for the conjugacy classes of $\mathbb{Z}_\ell^w \rtimes \mathfrak{S}_{\boldsymbol{\mu}}$ such that x_i and x are conjugate in $G_{\ell,w}$. However, as explained in Remark 3.1, the cycle structure of x_i in $G_{\ell,w}$ lies in $\mathcal{C}_\emptyset$. Then, for each $i \in I$, there is $\eta_i \vdash w$ such that the cycle structure of x_i in $G_{\ell,w}$ is $(\eta_i, \emptyset, \ldots, \emptyset)$. Hence, we have

$$\left(\widehat{\phi}_{\boldsymbol{\mu}} \otimes (\chi_{\mu_1} \boxtimes \cdots \boxtimes \chi_{\mu_\ell})\right)(x_i) = (\chi_{\mu_1} \boxtimes \cdots \boxtimes \chi_{\mu_\ell})(\eta_i),$$

because $\widehat{\phi}_{\boldsymbol{\mu}}(1) = \phi_{\boldsymbol{\mu}}(1) = 1$. Moreover, [2, 4.1] implies that the elements x_i are conjugate in $\mathbb{Z}_\ell^w \rtimes \mathfrak{S}_w$ if and only if the elements η_i are conjugate in $\mathfrak{S}_w$. We then deduce that the elements $\{\eta_i, i \in I\}$ form a system of representatives for the conjugacy classes of $\mathfrak{S}_{\boldsymbol{\mu}}$ such that π and η_i are conjugate in $\mathfrak{S}_w$. It then follows that

$$\begin{aligned} \theta_{\boldsymbol{\mu}}(x) &= \ell^w |\,\mathrm{C}_{\mathfrak{S}_w}(\pi)| \sum_{i \in I} \frac{1}{\ell^w |\,\mathrm{C}_{\mathfrak{S}_{\boldsymbol{\mu}}}(\eta_i)|} (\chi_{\mu_1} \boxtimes \cdots \boxtimes \chi_{\mu_\ell})(\eta_i) \\ &= \Gamma_{\boldsymbol{\mu}}(\pi). \end{aligned}$$

□

3.2. **A $\mathbb{Z}$-basis of the character ring of $\mathfrak{S}_{2w}$.** Fix a positive integer w. In this section, we will construct a new $\mathbb{Z}$-basis of the character ring of $\mathfrak{S}_{2w}$.

For $\mu \vdash w$, we define

$$\gamma_\mu = \operatorname{Ind}_{\mathfrak{S}_w \times \mathfrak{S}_w}^{\mathfrak{S}_{2w}} (\chi_\mu \boxtimes \chi_\mu). \tag{5}$$

We put

$$B_w = \{\gamma_\mu \,|\, \mu \vdash w\} \cup \{\chi_\lambda \,|\, \lambda \notin \mathcal{P}'_{2w}\},$$

where P'_{2w} is the set of partitions defined in Theorem 1.2.

Proposition 3.3. *The set B_w is a $\mathbb{Z}$-basis of the ring $\mathbb{Z}\operatorname{Irr}(\mathfrak{S}_{2w})$.*

Proof. For $\mu = (\mu_1, \ldots, \mu_r) \vdash w$, we define

$$\widetilde{\mu} = (2\mu_1, \ldots, 2\mu_r).$$

Note that the map $\mu \mapsto \widetilde{\mu}$ is a bijection between $\mathcal{P}_w$ and $\mathcal{P}'_{2w}$. It follows that $|B_w| = |\operatorname{Irr}(\mathfrak{S}_{2w})|$. Then, to prove that B_w is a $\mathbb{Z}$-basis of $\mathbb{Z}\operatorname{Irr}(\mathfrak{S}_{2w})$, it is sufficient to prove that B_w is a generating family over $\mathbb{Z}$. Let $\mu \vdash w$. Then we have

$$\gamma_\mu = \sum_{\lambda \vdash 2w} c^\lambda_{\mu\mu} \chi_\lambda,$$

where, for each $\lambda \vdash 2w$, $c^\lambda_{\mu\mu}$ is the Littelwood-Richardson coefficient associated to the partitions μ, μ and λ. If we arrange the partitions of w in lexicographic order, then [6, 6.1.2] implies that the matrix

$$P = (c^{\widetilde{\mu}}_{\mu\mu})_{\mu \vdash w}$$

is a lower triangular matrix with diagonal entries equal to 1. In particular, P is invertible over $\mathbb{Z}$. Then, using P^{-1} we can write the characters $\chi_{\widetilde{\mu}}$ for $\mu \vdash w$ as a linear combination of elements of B_w with coefficients in $\mathbb{Z}$. This yields the claim. □

3.3. **A generalized perfect isometry.** First, we will briefly present the notion of generalized perfect isometry, introduced in [9] by Külshammer, Olsson and Robinson. For a union $\mathcal{C}$ of conjugacy classes of a finite group G, we say that two irreducible characters α and β are *orthogonal across* $\mathcal{C}$ if

$$\langle\, \alpha, \beta \,\rangle_{\mathcal{C}} := \frac{1}{|G|} \sum_{g \in \mathcal{C}} \alpha(g)\overline{\beta(g)} = 0.$$

Then, we define the *$\mathcal{C}$-blocks* of G to be the minimal non-empty subsets of $\operatorname{Irr}(G)$ subject to being orthogonal across $\mathcal{C}$. For $b \subseteq \operatorname{Irr}(G)$, we write $(b, \mathcal{C})$ to indicate that b is a union of $\mathcal{C}$-blocks. Then a bijection $\mathcal{I} : b \to b'$ is a *generalized perfect isometry* (with respect to $\mathcal{C}$ and $\mathcal{C}'$ between two unions of blocks $(b, \mathcal{C})$ and $(b', \mathcal{C}')$ of G and H, if there are signs $\{\eta(\alpha) \,|\, \alpha \in b\}$ such that, for all α, $\beta \in b$,

$$\langle\, \alpha, \beta \,\rangle_{\mathcal{C}} = \langle\, \eta(\alpha)\mathcal{I}(\alpha), \eta(\beta)\mathcal{I}(\beta) \,\rangle_{\mathcal{C}'}.$$

Let w be a positive integer. In the following, we put

$$G_w = \mathbb{Z}_2 \wr \mathfrak{S}_w.$$

Note that, with the notation of Section 3.1, we have $G_w = G_{2,w}$. We can then apply the results of Section 3.1 to G_w. In particular, the irreducible characters of G_w are labelled by the set $\mathcal{MP}_{2,w}$ and the character corresponding to $\boldsymbol{\mu} \in \mathcal{MP}_{2,w}$

is denoted by $\theta_{\boldsymbol{\mu}}$. We also denote by $\mathcal{C}_\emptyset$ the set of elements of G_w with cycle structure $(\mu_1, \emptyset)$ for some $\mu_1 \vdash w$.

In order to describe [2, 3.6] for $p = 2$, we introduce a bijection on $\mathcal{MP}_{2,w}$ (denoted by $\check{}$) defined by

$$\check{\boldsymbol{\mu}} = (\mu_1, \mu_2^*) \quad \text{for } \boldsymbol{\mu} = (\mu_1, \mu_2).$$

Remark 3.4. *This definition comes from* [7]*, on which* [2, 3.6] *is based, and counter-balances the appearance in the Murnaghan-Nakayama Formula for G_w of some negative signs, coming from the fact that $\psi_2(-1) = -1$.*

In order to give the main result of this section, we have to recall some definitions. For $\lambda \vdash n$, we denote by $\gamma(\lambda)$ the 2-core of λ ([8, 2.7]). Recall that two characters χ_{λ_1} and χ_{λ_2} are in the same 2-block of $\mathfrak{S}_n$ if and only if the partitions λ_1 and λ_2 have the same 2-core. Then the set of 2-blocks of $\mathfrak{S}_n$ can be labelled by the set of 2-cores of $\mathfrak{S}_n$. For $\lambda \vdash n$, the integer $w = \frac{1}{2}(n - |\gamma(\lambda)|)$ is called the 2-weight of λ. Since irreducible characters of $\mathfrak{S}_n$ lying in the same 2-block have the same 2-core, it follows that the weight is an invariant of the block. We can thus define the weight of a block b as the weight of all characters in b. We now can state:

Theorem 3.5. *(cf.* [2, 3.6]*) Let n be a positive integer. Let b be a 2-block of $\mathfrak{S}_w$ of 2-weight $w \neq 0$. For $\chi_\lambda \in b$, write $\alpha_\lambda \in \mathcal{MP}_{2,w}$ the 2-quotient of λ. We can associate to α_λ the character θ_{α_λ} defined in Formula (2). Then the map*

$$\mathcal{J} : b \to \mathrm{Irr}(G_w), \ \chi_\lambda \mapsto \theta_{\check{\alpha_\lambda}}$$

is a generalized perfect isometry between $(b, 2\text{-reg})$ and $(\mathrm{Irr}(G_w), \mathcal{C}_\emptyset)$.

3.4. **Proof of Theorem 1.2.** We keep the notation of the above sections. Recall that, by [2, 3.1], it is possible to define 2-quotients of partitions in such a way that, for any n and any $\lambda \vdash n$, if $\alpha_\lambda = (\lambda^{(1)}, \lambda^{(2)})$, then $\alpha_{\lambda^*} = (\lambda^{(2)^*}, \lambda^{(1)^*})$. Moreover, note that if χ_λ lies in a 2-block of $\mathfrak{S}_n$ with 2-weight w, then $|\lambda^{(1)}| + |\lambda^{(2)}| = w$.

Lemma 3.6. *Let n be a positive integer and $\lambda \vdash n$. Then λ is self-conjugate if and only if its 2-quotient has the form (μ, μ^*). In particular, if w is the 2-weight of λ, then w has to be even.*

Proof. Since $\alpha_{\lambda^*} = (\lambda^{(2)^*}, \lambda^{(1)^*})$, it immediately follows that $\alpha_\lambda = (\lambda^{(1)}, \lambda^{(1)^*})$ whenever $\lambda \vdash n$ is a self-conjugate partition. □

Using this, we can reduce Theorem 1.2 to the same question on 2-blocks of $\mathfrak{S}_n$ with even weight. More precisely, we have

Lemma 3.7. *The symmetric group $\mathfrak{S}_n$ has a 2-basic set containing every χ_λ with $\lambda = \lambda^*$ if and only if every 2-block b of $\mathfrak{S}_n$ with even weight w has a 2-basic set containing all $\chi_\lambda \in b$ with $\lambda = \lambda^*$.*

Proof. Using [2, 2.1], we can reduce the problem to the 2-blocks of $\mathfrak{S}_n$. Let b be a 2-block of $\mathfrak{S}_n$ with odd weight. Lemma 3.6 implies that if $\chi_\lambda \in b$, then $\lambda \neq \lambda^*$. Therefore, it is sufficient to prove that b has a 2-basic set. For this, we use Theorem 3.5, which implies that $\mathcal{J} : b \to \mathrm{Irr}(G_w)$ is a generalized perfect isometry between $(b, 2\text{-reg})$ and $(\mathrm{Irr}(G_w), \mathcal{C}_\emptyset)$. Moreover, if we denote by $B_\emptyset$ the set of irreducible characters of G_w labelled by elements of $\mathcal{C}_\emptyset$, then Lemma [2, 4.2] implies $B_\emptyset$ is a $\mathcal{C}_\emptyset$-basic set of G_w. The result then follows from [2, 2.2]. □

The case of blocks of 2-weight 0 is easy to deal with. Such a block b consists of a unique irreducible character χ_λ of $\mathfrak{S}_n$, such that λ is a self-conjugate partition (since it is its own 2-core, and one shows easily that any 2-core must be self-conjugate). Hence $\{\chi_\lambda\}$ is a 2-basic set for b, which obviously satisfies the required property. We next solve the case of blocks with positive 2-weight.

Proposition 3.8. *Let b be a 2-block of $\mathfrak{S}_n$ with even weight $2w$ for some positive integer w. Then b has a 2-basic set containing all irreducible characters of b labelled by self-conjugate partitions.*

Proof. Fix a 2-block b of $\mathfrak{S}_n$ with even 2-weight $2w$. Theorem 3.5 implies that $\mathcal{J} : b \to \mathrm{Irr}(G_{2w})$ is a generalized perfect isometry between $(b, 2\text{-reg})$ and $(\mathrm{Irr}(G_{2w}), \mathcal{C}_\emptyset)$. We parametrize the irreducible characters of G_{2w} by the elements of $\mathcal{MP}_{2,2w}$ as described above, and the character corresponding to $\boldsymbol{\mu} \in \mathcal{MP}_{2,2w}$ will be denoted by $\theta_{\boldsymbol{\mu}}$ as in Equation (3). Let $\chi_\lambda \in b$ with $\lambda = \lambda^*$. Then, Lemma 3.6 and the definition of $\mathcal{J}$ imply that

$$\mathcal{J}(\chi_\lambda) = \theta_{(\mu,\mu)}.$$

Therefore, using [2, 2.2], we see that proving that b has a 2-basic set containing all irreducible characters of b labelled by self-conjugate partitions is equivalent to showing that the group G_{2w} has a $\mathcal{C}_\emptyset$-basic set containing all irreducible characters of G_{2w} labelled by bi-partitions of the form (μ, μ).

Let $B_\emptyset$ be the set of irreducible characters of G_{2w} labelled by elements of $\mathcal{C}_\emptyset$. More precisely, the characters of $B_\emptyset$ are the characters of G_{2w} with $\mathbb{Z}_2^{2w}$ in their kernel. As we mentioned in Lemma 3.7, $B_\emptyset$ is a $\mathcal{C}_\emptyset$-basic set of G_{2w}. Note that, for all λ, $\pi \vdash 2w$, we have

$$\theta_{(\lambda,\emptyset)}(\pi, \emptyset) = \chi_\lambda(\pi). \tag{6}$$

However, $B_\emptyset$ doesn't have the required property. We will now construct from $B_\emptyset$ a $\mathcal{C}_\emptyset$-basic set containing the set of characters $\{\theta_{(\mu,\mu)} \mid \mu \vdash w\}$. Proposition 3.2 implies that

$$\theta_{(\mu,\mu)}(\pi, \emptyset) = \Gamma_{(\mu,\mu)}(\pi). \tag{7}$$

Note that $\Gamma_{(\mu,\mu)}$ is the character γ_μ defined in Formula (5). Furthermore, Formulae (6) and (7) imply that, if we can find a $\mathbb{Z}$-basis of $\mathfrak{S}_{2w}$ containing the character γ_μ for every $\mu \vdash w$ and the irreducible characters χ_λ for λ in some parametrizing set I, then the set of irreducible characters of G_w labelled by $\{(\mu, \mu),\ \mu \vdash w\} \cup \{(\lambda, \emptyset),\ \lambda \in I\}$ is a $\mathcal{C}_\emptyset$-basic set of G_{2w}. We therefore get the desired result, with $I = \mathcal{P}_{2w} \backslash \mathcal{P}'_{2w}$, by Proposition 3.3. $\square$

Remark 3.9. *Note that the generalized perfect isometry $\mathcal{J}$ of Theorem 3.5 is not one of the isometry described by Osima between b and $\mathrm{Irr}(G_w)$. Indeed, Osima's isometry is a generalized perfect isometry between $(b, 2\text{-reg})$ and $(\mathrm{Irr}(G_w), \mathcal{D}_\emptyset)$ where $\mathcal{D}_\emptyset$ is the set of elements of cycle structure $(\mu_1, \mu_2) \in \mathcal{MP}_{2,w}$ with $\mu_1 = \emptyset$. It seems to be more difficult to prove a result similar to Proposition 3.8 for $(\mathrm{Irr}(G_{2w}), \mathcal{D}_\emptyset)$.*

We now can prove Theorem 1.2. As explained in the proof of Lemma 3.7, it is sufficient to construct a 2-basic set with the required property for all 2-blocks of $\mathfrak{S}_n$. Let b be a 2-block of $\mathfrak{S}_n$ with weight w.

If w is odd, then we choose the 2-basic set B_b of b constructed in the proof of Lemma 3.7. In this case, $\mathcal{P}'_w$ is empty and the 2-quotients of the characters in B_b have the form $(\mu, \emptyset)$ with $\mu \vdash w$.

If w is odd, then Proposition 3.8 implies that b has a 2-basic set B_b satisfying $\chi_\lambda \in B_b$ if and only if $\alpha_\lambda = (\mu, \emptyset)$ with $\mu \in \mathcal{P}_w \backslash \mathcal{P}'_w$ or $\alpha_\lambda = (\mu, \mu^*)$ for $\mu \vdash w/2$.

Thus, the set $\mathcal{B}_\Lambda$ defined in Theorem 1.2 is a 2-basic set of $\mathfrak{S}_n$.

References

[1] O. Brunat. Basic sets in defining characteristic for general linear groups of small rank. *J. Pure Appl. Algebra*, (213):698–710, 2009.

[2] O. Brunat and J.B. Gramain. A basic set for the alternating group. *To appear in J. Reine Angew. Math.*

[3] M. Geck. Basic sets of Brauer characters of finite groups of Lie type. II. *J. London Math. Soc. (2)*, 47(2):255–268, 1993.

[4] M. Geck. Basic sets of Brauer characters of finite groups of Lie type. III. *Manuscripta Math.*, 85(2):195–216, 1994.

[5] M. Geck and G. Hiss. Basic sets of Brauer characters of finite groups of Lie type. *J. Reine Angew. Math.*, 418:173–188, 1991.

[6] M. Geck and G. Pfeiffer. *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, volume 21 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 2000.

[7] J.-B. Gramain. On defect groups for generalized blocks of the symmetric group. *J. London Math. Soc.*, 78(1):155–171, 2008.

[8] G. James and A. Kerber. *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981.

[9] B. Külshammer, J. Olsson, and G.R. Robinson. Generalized blocks for symmetric groups. *Invent. Math.*, 151(3):513–552, 2003.

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